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Laplacian eigenvalue functionals and metric deformations on compact manifolds

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Abstract

In this paper, we investigate critical points of the eigenvalues of the Laplace operator considered as functionals on the space of Riemannian metrics or a conformal class of metrics on a compact manifold. We introduce a natural notion of the critical metric of such a functional and obtain necessary and sufficient conditions for a metric to be critical. We derive specific consequences concerning possible locally maximizing metrics. We also characterize critical metrics of the ratio of two consecutive eigenvalues.

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1. Introduction

The eigenvalues of the Laplace–Beltrami operator associated with a Riemannian metric on a closed manifold are among the most natural global Riemannian invariants defined independently from curvature. One of the main topics in spectral geometry is the study of uniform boundedness of eigenvalues under some constraints and the finding of eventual extremal metrics. Let us start by recalling some important results in this direction, where the eigenvalues are considered as functionals on the set of Riemannian metrics of fixed volume.

In all the sequel, we will denote by M a compact smooth manifold of dimension $n \geq 2$ and, for any Riemannian metric g on M, by

$$0 < \lambda_1(g) \leq \cdots \leq \lambda_k(g) \leq \cdots \to \infty$$

the sequence of eigenvalues of the Laplacian Δ_g associated with g, repeated according to their multiplicities. Notice that λ_k is not invariant under scaling (i.e., for all $c > 0$, $\lambda_k(cg) = \frac{1}{c}\lambda_k(g)$). Hence, a normalization is needed and it is common to restrict the functional λ_k to the set $\mathcal{R}(M)$ of Riemannian metrics of fixed volume. We will denote by $C(g)$ the set of Riemannian metrics conformal to g and having the same volume as g.

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The result that in dimension 2, for any $k \geq 1$, the functional λ_k is uniformly bounded on the set of metrics of fixed area is due to Korevaar [24], after having been proved for $k = 1$ by Hersch [21] in the case of the 2-sphere $\mathbb{S}^2$, and by Yang and Yau [34] and Li and Yau [25] for all compact surfaces (see also [11] for an improvement of the Yang–Yau upper bound of λ_1 in terms of the genus).

The situation differs in dimension $n \geq 3$. Indeed, on the basis of earlier results on spheres obtained by many authors [31,26,3,33,27], Colbois and Dodziuk [5] proved that, for any compact manifold M of dimension $n \geq 3$, the functional λ_1 is unbounded on $\mathcal{R}(M)$.

However, for all $k \geq 1$, the functional λ_k becomes uniformly bounded when restricted to a conformal class of metrics of fixed volume $C(g)$. This result was first proved for $k = 1$ by the authors [12] (see also [17]) and, for any $k \geq 1$, by Korevaar [24] (see also [20]).

Existence results for maximizing metrics are available for only few situations and concern exclusively the first eigenvalue functional. Hersch [21] proved that the standard metric is the only maximizing metric for λ_1 on the 2-sphere $\mathbb{S}^2$. The same result holds for the standard metric of the real projective plane $\mathbb{R}P^2$ (Li and Yau [25]). Nadirashvili [28] outlined a proof for the existence of maximizing metrics for λ_1 on the 2-torus and the Klein bottle (see [19,18] for additional details on Nadirashvili's paper).

In higher dimensions, the authors [12] gave a sufficient condition for a Riemannian metric g to maximize λ_1 in its conformal class $C(g)$. This condition is fulfilled in particular by the standard metric of the sphere $\mathbb{S}^n$ (which enabled us to solve Berger's problem concerning the maximization of λ_1 restricted to the standard conformal class of $\mathbb{S}^n$), and more generally, by the standard metric of any compact rank-1 symmetric space. The flat metrics g_{sq} and g_{eq} on the 2-torus $\mathbb{T}^2$ associated with the square lattice $\mathbb{Z}^2$ and the equilateral lattice $\mathbb{Z}(1,0) \oplus \mathbb{Z}(1/2, \sqrt{3}/2)$, respectively, are also maximizing metrics of λ_1 in their conformal classes (see [12,25]).

In this paper we address the following natural questions:

(1) *What about critical points of the functional $g \mapsto \lambda_k(g)$?*
(2) *How can one deform a Riemannian metric g in order to increase, or decrease, the k-th eigenvalue λ_k?*

Despite the non-differentiability of the functional λ_k with respect to metric deformations, perturbation theory enables us to prove that, for any analytic deformation g_t of a metric g, the function $t \mapsto \lambda_k(g_t)$ always admits left and right derivatives at $t = 0$ (see Theorem 2.1(i) below). Moreover, these derivatives can be expressed in terms of the eigenvalues of the following quadratic form:

$$Q_h(u) = -\int_M \left(\mathrm{d}u \otimes \mathrm{d}u + \frac{1}{4}\Delta_g u^2 g, h\right) v_g,$$

with $h = \frac{\mathrm{d}}{\mathrm{d}t} g_t|_{t=0}$, restricted to the eigenspace $E_k(g)$ (Theorem 2.1 and Lemma 2.1). This allows us to give partial answers to Question (2) above (Corollary 2.1). In particular, if Q_h is positive definite on $E_k(g)$, then $\lambda_k(g_{-t}) < \lambda_k(g) < \lambda_k(g_t)$ for all $t \in (0, \varepsilon)$, for some positive ε.

As regards Question (1), the existence of left and right derivatives for $t \mapsto \lambda_k(g_t)$, suggests the following natural definition of criticality. Indeed, a metric g will be termed *critical* for the functional λ_k if, for any volume-preserving deformation g_t of g, one has

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\right|_{t=0^-} \times \left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\right|_{t=0^+} \leq 0;$$

this means that either

$$\lambda_k(g_t) \leq \lambda_k(g) + o(t) \quad \text{or} \quad \lambda_k(g_t) \geq \lambda_k(g) + o(t).$$

It is clear that if g is a locally maximizing or a locally minimizing metric of λ_k, then g is a critical metric for λ_k in the previous sense.

In earlier work [13,15], we treated the particular case $k = 1$ and gave a necessary condition [13, Theorem 1.1] and [15, Theorem 2.1] as well as a sufficient condition [13, Proposition 1.1] and [15, Theorem 2.2] for a metric g to be critical for the functional λ_1 or for the restriction of the λ_1 to a conformal class.

In Sections 3 and 4 below, we extend the results of [13,15] to higher order eigenvalues, and weaken the sufficient conditions given there. Actually, as we will see, in many cases, the necessary condition of criticality is also sufficient (Theorems 3.1 and 4.1). Given a metric g on M, we will prove that

- *Necessary conditions* (Theorem 3.1(i) and Theorem 4.1(i)):
  (i) If g is critical for the functional λ_k, then there exists a finite family $\{u_1, \ldots, u_d\}$ of eigenfunctions associated with λ_k such that
  $$\sum_{i \le d} \mathrm{d}u_i \otimes \mathrm{d}u_i = g.$$
  (ii) If g is a critical metric of the functional λ_k restricted to the conformal class of g, then there exists a finite family $\{u_1, \ldots, u_d\}$ of eigenfunctions associated with λ_k such that
  $$\sum_{i \le d} u_i^2 = 1.$$
- *Sufficient conditions* (Theorem 3.1(ii) and Theorem 4.1(ii)): If $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$ (which means that $\lambda_k(g)$ corresponds to the first one or the last one of a cluster of equal eigenvalues), then the necessary conditions above are also sufficient.

The condition (i) above means that the map $u := (u_1, \ldots, u_d) : (M, g) \longrightarrow \mathbb{R}^d$, is an isometric immersion, whose image is a minimally immersed submanifold of the Euclidean sphere $\mathbb{S}^{d-1}(\sqrt{\frac{n}{\lambda_k(g)}})$ of radius $\sqrt{\frac{n}{\lambda_k(g)}}$ (see [30]). In other words, a metric g is critical for the functional λ_k, for some $k \ge 1$, if and only if g is induced on M by a minimal immersion of M into a sphere. Therefore, the classification of critical metrics of eigenvalue functionals on M reduces to the classification of minimal immersions of M into spheres. The many existence and classification results (see for instance [16,22,9,32] and the references therein) of minimal immersions into spheres give examples of critical metrics for the eigenvalue functionals.

Notice that, for the first eigenvalue functional, the critical metrics are classified on surfaces of genus 0 and 1. Indeed, on $\mathbb{S}^2$ and $\mathbb{R}P^2$, the standard metrics are the only critical ones [12]. On the torus $\mathbb{T}^2$, the flat metrics g_{eq} and g_{sq} mentioned above are, up to dilatations, the only critical metrics for λ_1 [13]. The metric g_{eq} corresponds to a maximizer for λ_1 [28], while g_{sq} is a saddle point. For the Klein bottle $\mathbb{K}$, Jakobson, Nadirashvili and Polterovich [23] showed the existence of a critical metric and El Soufi, Giacomini and Jazar [10] proved that this metric is, up to dilatations, the unique critical metric for λ_1 on $\mathbb{K}$.

Now, the condition (ii), concerning the criticality for λ_k restricted to a conformal class, is equivalent to the fact that the map $u := (u_1, \ldots, u_d) : (M, g) \longrightarrow \mathbb{S}^{d-1}$ is a harmonic map with energy density $e(u) = \frac{\lambda_k(g)}{2}$ (see for instance [8]). Thus, a metric g is critical for some λ_k restricted to the conformal class of g if and only if (M, g) admits a harmonic map of constant energy density in a sphere. In particular, the metric of any homogeneous compact Riemannian space is critical for λ_k restricted to its conformal class (for other examples see [29,9,32] and the references therein).

A consequence of the necessary condition (ii) is that, if g is a critical metric of λ_k restricted to $C(g)$, then the multiplicity of $\lambda_k(g)$ is at least 2 (Corollary 4.2). This means that $\lambda_k(g) = \lambda_{k-1}(g)$ or $\lambda_k(g) = \lambda_{k+1}(g)$. In the case where the metric g is a local maximizer of λ_k restricted to $C(g)$, we prove that one necessarily has $\lambda_k(g) = \lambda_{k+1}(g)$ (Corollary 4.2). For a local minimizer, one has $\lambda_k(g) = \lambda_{k-1}(g)$. Together with a recent result of Colbois and the first author [6], this result tells us that a Riemannian metric can never maximize two consecutive eigenvalues simultaneously on its conformal class (Corollary 4.3). In fact, if g maximizes λ_k on $C(g)$, then

$$\lambda_{k+1}(g)^{\frac{n}{2}} \le \sup_{g' \in C(g)} \lambda_{k+1}(g')^{\frac{n}{2}} - n^{\frac{n}{2}} \omega_n,$$

where ω_n is the volume of the unit Euclidean n-sphere.

As an application of the results above, one can derive characterizations of the metrics which are critical for various functions of eigenvalues. To illustrate this, we treat in the last section of this paper the case of the ratio functional $\frac{\lambda_{k+1}}{\lambda_k}$ of two consecutive eigenvalues and give characterizations of critical metrics for these functionals.

We end this section by pointing out that, in order to compare the eigenvalues of the Dirac and Laplace operators on two-dimensional tori, Agricola, Ammann and Friedrich [1] have also investigated the variation of such eigenvalues up to order 4.

2. Derivatives of eigenvalues with respect to metric deformations

Let M be a compact smooth manifold of dimension $n \geq 2$. For any Riemannian metric g on M, we denote by $0 < \lambda_1(g) \leq \lambda_2(g) \cdots \leq \lambda_k(g) \leq \cdots$ the eigenvalues of the Laplace–Beltrami operator Δ_g associated with g. For any $k \in \mathbb{N}$, we denote by $E_k(g) = \mathrm{Ker}(\Delta_g - \lambda_k(g)I)$ the eigenspace corresponding to $\lambda_k(g)$ and by $\Pi_k : L^2(M, g) \longrightarrow E_k(g)$ the orthogonal projection on $E_k(g)$.

Let us fix a positive integer k and consider the functional $g \longmapsto \lambda_k(g)$. This functional is continuous but not differentiable in general. However, perturbation theory tells us that λ_k is left and right differentiable along any analytic curve of metrics. The main purpose of this section is to express the derivatives of λ_k with respect to analytic metric deformations, in terms of the eigenvalues of an explicit quadratic form on $E_k(g)$. Indeed, we have the following:

Theorem 2.1. *Let g be a Riemannian metric on M and let $(g_t)_t$ be a family of Riemannian metrics analytically indexed by $t \in (-\epsilon, \epsilon)$, such that $g_0 = g$. The following hold:*

(i) *The function $t \in (-\epsilon, \epsilon) \longmapsto \lambda_k(g_t)$ admits left and right derivatives at $t = 0$.*
(ii) *The derivatives $\frac{d}{dt}\lambda_k(g_t)|_{t=0^-}$ and $\frac{d}{dt}\lambda_k(g_t)|_{t=0^+}$ are eigenvalues of the operator $\Pi_k \circ \Delta' : E_k(g) \longrightarrow E_k(g)$, where $\Delta' = \frac{d}{dt}\Delta_{g_t}|_{t=0}$.*
(iii) *If $\lambda_k(g) > \lambda_{k-1}(g)$, then $\frac{d}{dt}\lambda_k(g_t)|_{t=0^-}$ and $\frac{d}{dt}\lambda_k(g_t)|_{t=0^+}$ are the greatest and the least eigenvalues of $\Pi_k \circ \Delta'$ on $E_k(g)$, respectively.*
(iv) *If $\lambda_k(g) < \lambda_{k+1}(g)$, then $\frac{d}{dt}\lambda_k(g_t)|_{t=0^-}$ and $\frac{d}{dt}\lambda_k(g_t)|_{t=0^+}$ are the least and the greatest eigenvalues of $\Pi_k \circ \Delta'$ on $E_k(g)$, respectively.*

Proof. The family of operators $\Delta_t := \Delta_{g_t}$ depends analytically on t and, for all t, Δ_t is self-adjoint with respect to the L^2 inner product induced by g_t (but not necessarily to that induced by g). However, as was done in [2], after a conjugation with the unitary isomorphism

$$U_t : L^2(M, g) \to L^2(M, g_t)$$
$$u \mapsto \left(\frac{|g|}{|g_t|}\right)^{1/4} u,$$

where $|g_t|$ is the Riemannian volume density of g_t, we obtain an analytic family $P_t = U_t^{-1} \circ \Delta_t \circ U_t$ of operators such that, for all $t \in (-\epsilon, \epsilon)$, P_t is self-adjoint with respect to the L^2 inner product induced by g. Moreover, P_t and Δ_t have the same spectrum. In particular, $\lambda_k(g_t)$ is an eigenvalue of P_t. The Rellich–Kato perturbation theory of unbounded self-adjoint operators applies to the analytic family of operators $t \mapsto P_t$. Therefore, if we denote by m the dimension of $E_k(g)$, then there exist, for all $t \in (-\epsilon, \epsilon)$, m eigenvalues $\Lambda_1(t), \ldots, \Lambda_1(t)$ of P_t associated with an $L^2(M, g)$-orthonormal family of eigenfunctions $v_1(t), \ldots, v_m(t)$ of P_t, that is $P_t v_i(t) = \Lambda_i(t) v_i(t)$, so that $\Lambda(0) = \cdots = \Lambda_m(0) = \lambda_k(g)$, and, for all $i \leq m$, both $\Lambda_i(t)$ and $v_i(t)$ depend analytically on t. Setting, for all $i \leq m$ and all $t \in (-\epsilon, \epsilon)$, $u_i(t) = U_t v_i(t)$, we get, for all $i \leq m$,

$$\Delta_t u_i(t) = \Lambda_i(t) u_i(t) \tag{1}$$

and the family $\{u_1(t), \ldots, u_m(t)\}$ is orthonormal in $L^2(M, g_t)$. Since $t \mapsto \lambda_k(t)$ is continuous and, for all $i \leq m$, $t \mapsto \Lambda_i(t)$ is analytic with $\Lambda_i(0) = \lambda_k(g)$, there exist $\delta > 0$ and two integers $p, q \leq m$ such that

$$\lambda_k(g_t) = \begin{cases} \Lambda_p(t) & \text{for } t \in (-\delta, 0) \\ \Lambda_q(t) & \text{for } t \in (0, \delta). \end{cases}$$

Assertion (i) follows immediately. Moreover, one has

$$\left.\frac{d}{dt}\lambda_k(t)\right|_{t=0^-} = \Lambda_p'(0)$$

and

$$\left.\frac{d}{dt}\lambda_k(t)\right|_{t=0^+} = \Lambda_q'(0).$$

Differentiating both sides of (1) at $t = 0$, we get

$$\Delta' u_i + \Delta u_i' = \Lambda_i'(0) u_i + \lambda_k(g) u_i'$$

with $u_i' = \frac{\mathrm{d}}{\mathrm{d}t} u_i(t)|_{t=0}$ and $u_i := u_i(0)$. Multiplying this last equation by u_j and integrating by parts with respect to the Riemannian volume element v_g of g, we obtain

$$\int_M u_j \Delta' u_i \, v_g = \begin{cases} \Lambda_i'(0) & \text{if } j = i \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

Since $\{u_1, \ldots, u_m\}$ is an orthonormal basis of $E_k(g)$ with respect to the L^2 inner product induced by g, we deduce that

$$(\Pi_k \circ \Delta') u_i = \Lambda_i'(0) u_i.$$

In particular, $\Lambda_p'(0)$ and $\Lambda_q'(0)$ are eigenvalues of $\Pi_k \circ \Delta'$, which proves Assertion (ii).

Assume now $\lambda_k(g) > \lambda_{k-1}(g)$. Hence, for all $i \le m$, $\Lambda_i(0) = \lambda_k(g) > \lambda_{k-1}(g)$. By continuity, we have $\Lambda_i(t) > \lambda_{k-1}(g_t)$ for sufficiently small t. Hence, there exists $\eta > 0$ such that, for all $t \in (-\eta, \eta)$ and all $i \le m$, $\Lambda_i(t) \ge \lambda_k(g_t)$, which means that $\lambda_k(g_t) = \min\{\Lambda_1(t), \ldots, \Lambda_m(t)\}$. This implies that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \lambda_k(g_t)\right|_{t=0^-} = \max\left\{\Lambda_1'(0), \ldots, \Lambda_m'(0)\right\}$$

and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \lambda_k(g_t)\right|_{t=0^+} = \min\left\{\Lambda_1'(0), \ldots, \Lambda_m'(0)\right\}.$$

Assertion (iii) is proved.

The proof of Assertion (iv) is similar. Indeed, if $\lambda_k(g) < \lambda_{k+1}(g)$, one has, for sufficiently small t, $\lambda_k(g_t) = \max\{\Lambda_1(t), \ldots, \Lambda_m(t)\}$ and, then,

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \lambda_k(g_t)\right|_{t=0^+} = \max\left\{\Lambda_1'(0), \ldots, \Lambda_m'(0)\right\}$$

and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t} \lambda_k(g_t)\right|_{t=0^-} = \min\left\{\Lambda_1'(0), \ldots, \Lambda_m'(0)\right\}. \quad \square$$

The quadratic form associated with the symmetric operator $\Pi_k \circ \Delta'$ acting on $E_k(g)$ can be expressed explicitly as follows:

Lemma 2.1. *Let $(g_t)_t$ be an analytic deformation of the metric g and let $h := \frac{\mathrm{d}}{\mathrm{d}t} g_t|_{t=0}$. The operator $P_{k,h} := \Pi_k \circ \Delta' : E_k(g) \to E_k(g)$ is symmetric with respect to the L^2-norm induced by g; the corresponding quadratic form on $E_k(g)$ is given by*

$$Q_h(u) := \int_M u P_{k,h} u v_g = -\int_M \left(\mathrm{d}u \otimes \mathrm{d}u + \frac{1}{4} \Delta_g u^2 g, h\right) v_g,$$

where $(\ ,\ )$ is the pointwise inner product induced by g on covariant 2-tensors. Moreover, if, for all $t \in (-\epsilon, \epsilon)$, $g_t = \alpha_t g$ is conformal to g, then $h = \varphi g$ with $\varphi = \frac{\mathrm{d}}{\mathrm{d}t} \alpha_t|_{t=0}$, and, for all $u \in E_k(g)$,

$$Q_h(u) = -\int_M \varphi \left(|\mathrm{d}u|^2 + \frac{n}{4} \Delta_g u^2\right) v_g = -\frac{n}{2} \int_M \varphi \left(\lambda_k(g) u^2 - \frac{n-2}{n} |\mathrm{d}u|^2\right) v_g.$$

Proof. The derivative at $t = 0$ of $t \mapsto \Delta_{g_t}$ is given by the formula (see [4])

$$\Delta' u := \left.\frac{\mathrm{d}}{\mathrm{d}t} \Delta_{g_t} u\right|_{t=0} = (D\mathrm{d}u, h) - \left(\mathrm{d}u, \delta h + \frac{1}{2} \mathrm{d}(\mathrm{trace}_g h)\right), \tag{3}$$

where D is the canonical covariant derivative induced by g. Thus,

$$\int_M u\Delta' u v_g = \int_M u(D\mathrm{d}u, h)v_g - \frac{1}{2}\int_M \left(\mathrm{d}u^2, \delta h + \frac{1}{2}\mathrm{d}(\mathrm{trace}_g h)\right) v_g. \tag{4}$$

One has

$$uD\mathrm{d}u = \frac{1}{2}D\mathrm{d}u^2 - \mathrm{d}u \otimes \mathrm{d}u.$$

Hence,

$$\int_M u\,(D\mathrm{d}u, h)\,v_g = \frac{1}{2}\int_M (D\mathrm{d}u^2, h)v_g - \int_M (\mathrm{d}u \otimes \mathrm{d}u, h)v_g.$$

Since δ is the adjoint of D w.r.t. the $L^2(g)$ inner product, we obtain

$$\int_M u\,(D\mathrm{d}u, h)\,v_g = \frac{1}{2}\int_M (\mathrm{d}u^2, \delta h)v_g - \int_M (\mathrm{d}u \otimes \mathrm{d}u, h)v_g.$$

On the other hand

$$\int_M (\mathrm{d}u^2, \mathrm{d}(\mathrm{trace}_g h))v_g = \int_M \Delta_g u^2 \,\mathrm{trace}_g h v_g = \int_M (\Delta_g u^2 g, h)v_g.$$

Replacing in (4) one immediately gets the desired identity.

A straightforward computation gives the expression of Q_h for conformal deformations. □

In relation to the question (2) of the introduction, we give the following result which is a direct consequence of Theorem 2.1 and Lemma 2.1.

Corollary 2.1. *Let $(g_t)_t$ be an analytic deformation of a Riemannian metric g on M and let Q_h be the associated quadratic form defined as in Lemma 2.1, with $h = \frac{\mathrm{d}}{\mathrm{d}t} g_t|_{t=0}$.*

(i) *If Q_h is positive definite on $E_k(g)$, then there exists $\varepsilon > 0$ such that $\lambda_k(g_{-t}) < \lambda_k(g) < \lambda_k(g_t)$ for all $t \in (0, \varepsilon)$.*
(ii) *Assume that $\lambda_k(g) > \lambda_{k-1}(g)$. If there exists $u \in E_k(g)$ such that $Q_h(u) < 0$, then $\lambda_k(g_t) < \lambda_k(g)$ for all $t \in (0, \varepsilon)$, for some $\varepsilon > 0$.*
(iii) *Assume that $\lambda_k(g) < \lambda_{k+1}(g)$. If there exists $u \in E_k(g)$ such that $Q_h(u) > 0$, then $\lambda_k(g_t) > \lambda_k(g)$ for all $t \in (0, \varepsilon)$, for some $\varepsilon > 0$.*

In particular, if $Q_h(u) > 0$ for a first eigenfunction u, then $\lambda_1(g_t) < \lambda_1(g)$ for sufficiently small positive t.

3. Critical metrics of the eigenvalue functionals

Let M be a closed manifold of dimension $n \geq 2$ and let k be a positive integer. Before introducing the notion of the critical metric of the functional λ_k we recall that this functional is not scaling invariant. Therefore, we will restrict λ_k to the set of metrics of given volume. In view of Theorem 2.1, a natural way to introduce the notion of critical metric is the following:

Definition 3.1. A metric g on M is said to be "critical" for the functional λ_k if, for any volume-preserving analytic deformation $(g_t)_t$ of g with $g_0 = g$, the left and the right derivatives of $\lambda_k(g_t)$ at $t = 0$ satisfy

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\right|_{t=0^-} \times \left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\right|_{t=0^+} \leq 0.$$

It is easy to see that

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\right|_{t=0^+} \leq 0 \leq \left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\right|_{t=0^-} \iff \lambda_k(t) \leq \lambda_k(0) + o(t)$$

and

$$\left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\right|_{t=0^-} \leq 0 \leq \left.\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\right|_{t=0^+} \iff \lambda_k(t) \geq \lambda_k(0) + o(t).$$

Therefore, g is critical for λ_k if, for any volume-preserving analytic deformation $(g_t)_t$ of g, one of the following inequalities holds:

$$\lambda_k(g_t) \leq \lambda_k(g) + o(t)$$

or

$$\lambda_k(g_t) \geq \lambda_k(g) + o(t).$$

Of course, if g is a local maximizer or a local minimizer of λ_k, then g is critical in the sense of the previous definition.

Throughout the sequel, we will denote by $S_0^2(M, g)$ the space of covariant 2-tensors h satisfying $\int_M \mathrm{trace}_g h v_g = \int_M (g, h) v_g = 0$, endowed with its natural L^2-norm induced by g.

Proposition 3.1. *If g is a critical metric for the functional λ_k on M, then, for all $h \in S_0^2(M, g)$, the quadratic form*

$$Q_h(u) = -\int_M \left(\mathrm{d}u \otimes \mathrm{d}u + \frac{1}{4}\Delta_g u^2 g, h\right) v_g$$

is indefinite on $E_k(g)$.

Proof. Let $h \in S_0^2(M, g)$. The deformation of g defined for small t by $g_t = [\frac{\mathrm{vol}(g)}{\mathrm{vol}(g+th)}]^{2/n}(g + th)$, where $\mathrm{vol}(g)$ is the Riemannian volume of (M, g), is volume-preserving and depends analytically on t with $\frac{\mathrm{d}}{\mathrm{d}t} g_t|_{t=0} = h$. Using Theorem 2.1, we see that, if g is critical, then the operator $P_{h,k}$ admits non-negative and non-positive eigenvalues on $E_k(g)$ which means that the quadratic form Q_h is indefinite (Lemma 2.1). □

In the case where $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$, one can show that the converse of Proposition 3.1 is also true. Indeed, we have the following:

Proposition 3.2. *Let g be a Riemannian metric on M such that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$. Then g is critical for the functional λ_k if and only if, for all $h \in S_0^2(M, g)$, the quadratic form Q_h is indefinite on $E_k(g)$.*

Proof. Let $(g_t)_t$ be an analytic volume-preserving deformation of g and let $h = \frac{\mathrm{d}}{\mathrm{d}t} g_t|_{t=0}$. Since $\mathrm{vol}(g_t)$ is constant with respect to t, the tensor h belongs to $S_0^2(M, g)$ (indeed, $\int_M (g, h) v_g = \frac{\mathrm{d}}{\mathrm{d}t}\mathrm{vol}(g_t)|_{t=0} = 0$). The indefiniteness of Q_h implies that the operator $P_{k,h} = \Pi_k \circ \Delta'$ admits both non-negative and non-positive eigenvalues on $E_k(g)$ (see Lemma 2.1). The result follows immediately from Theorem 2.1(iii) and (iv). □

The indefiniteness of Q_h on $E_k(g)$ for all $h \in S_0^2(M, g)$ can be interpreted intrinsically in terms of the eigenfunctions of $\lambda_k(g)$ as follows.

Lemma 3.1. *Let g be a Riemannian metric on M. The two following conditions are equivalent:*

(i) *For all $h \in S_0^2(M, g)$, the quadratic form Q_h is indefinite on $E_k(g)$.*
(ii) *There exists a finite family $\{u_1, \ldots, u_d\} \subset E_k(g)$ of eigenfunctions associated with $\lambda_k(g)$ such that*

$$\sum_{i \leq d} \mathrm{d}u_i \otimes \mathrm{d}u_i = g.$$

Proof. The proof of "(i) implies (ii)" uses the same arguments as in the proof of Theorem 1.1 of [13]. For the sake of completeness, we will recall the main steps. First, we introduce the convex set $K \subset S^2(M, g)$ given by

$$K = \left\{ \sum_{j \in J} \left[\mathrm{d}u_j \otimes \mathrm{d}u_j + \frac{1}{4}\Delta_g u_j^2 g\right]; u_j \in E_k(g), J \subset \mathbb{N}, J \text{ finite} \right\}.$$

Let us first show that $g \in K$. Indeed, if $g \notin K$, then, applying the classical separation theorem in the finite dimensional subspace of $S^2(M, g)$ generated by K and g, endowed with the L^2 inner product induced by g, we deduce the existence of a 2-tensor $h \in S^2(M, g)$ such that $\int_M (g, h) v_g > 0$ and, for all $T \in K$, $\int_M (T, h) v_g \leq 0$. The tensor

$$h_0 = h - \left(\frac{1}{n\ \mathrm{vol}(g)} \int_M (g, h) v_g\right) g$$

belongs to $S_0^2(M,g)$ and we have, for all $u \in E_k(g)$, $u \neq 0$,

$$Q_{h_0}(u) = -\int_M \left(\mathrm{d}u \otimes \mathrm{d}u + \frac{1}{4}\Delta_g u^2 g, h \right) v_g + \frac{\int_M (g,h) v_g}{n\,\mathrm{vol}(g)} \int_M |\mathrm{d}u|^2 v_g$$
$$\geq \frac{\lambda_k(g)}{n\,\mathrm{vol}(g)} \int_M (g,h) v_g \int_M u^2 v_g.$$

Since $\int_M (g,h) v_g > 0$, the quadratic form Q_{h_0} is positive definite, which contradicts the assumption (i). Now, $g \in K$ means that there exists $u_1, \ldots, u_m \in E_k(g)$ such that

$$\sum_{i \leq d} \left(\mathrm{d}u_i \otimes \mathrm{d}u_i + \frac{1}{4}\Delta_g u_i^2 \right) g = g. \tag{5}$$

Hence, since $\Delta u_i^2 = 2(\lambda_k(g) u_i^2 - |\mathrm{d}u_i|^2)$, we obtain after taking the trace in (5),

$$\frac{\lambda_k(g)}{2} \sum_{i \leq d} u_i^2 = 1 + \frac{n-2}{2n} \sum_{i \leq d} |\mathrm{d}u_i|^2.$$

For $n = 2$, we immediately get $\sum_{i \leq d} u_i^2 = \frac{2}{\lambda_k(g)}$ and, for $n \geq 3$, we consider the function $f := \sum_{i \leq d} u_i^2 - \frac{n}{\lambda_k(g)}$ and observe that it satisfies

$$(n-2)\Delta_g f = 2(n-2) \left(\lambda_k(g) \sum_{i \leq d} u_i^2 - \sum_{i \leq d} |\mathrm{d}u_i|^2 \right) = -4\lambda_k(g) f.$$

Thus, $f = 0$ (the Laplacian being a non-negative operator) and, then, for all $n \geq 2$, $\sum_{i \leq d} u_i^2 = \frac{n}{\lambda_k(g)}$. Replacing in (5), we obtain

$$\sum_{i \leq d} \mathrm{d}u_i \otimes \mathrm{d}u_i = g.$$

Conversely, let $u_1, \ldots, u_d$ be as in (ii). This means that the map $x \in M \longmapsto u(x) = (u_1(x), \ldots, u_d(x)) \in \mathbb{R}^d$ is an isometric immersion. The vector $\Delta u(x) = (\Delta u_1(x), \ldots, \Delta u_d(x)) = \lambda_k(g) u(x)$ represents the mean curvature vector-field of the immersed submanifold $u(M)$. Hence, for all $x \in M$, the position vector $u(x)$ is normal to $u(M)$ which implies that $u(M)$ is contained in a sphere of $\mathbb{R}^d$ centered at the origin. Thus, $\sum_{i \leq d} u_i^2$ is constant on M. Consequently, for all $h \in S_0^2(M,g)$,

$$\sum_{i \leq d} Q_h(u_i) = \cdots = 0.$$

It follows that Q_h is indefinite on $E_k(g)$. □

Proposition 3.1, 3.2 and Lemma 3.1 lead to the following characterization of critical metrics of λ_k:

Theorem 3.1. *Let g be a Riemannian metric on M.*

(i) *If g is a critical metric of the functional λ_k, then there exists a finite family $\{u_1, \ldots, u_d\} \subset E_k(g)$ of eigenfunctions associated with $\lambda_k(g)$ such that*

$$\sum_{i \leq d} \mathrm{d}u_i \otimes \mathrm{d}u_i = g.$$

(ii) *Assume that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$. Then g is a critical metric of the functional λ_k if and only if there exists a finite family $\{u_1, \ldots, u_d\} \subset E_k(g)$ of eigenfunctions associated with $\lambda_k(g)$ such that*

$$\sum_{i \leq d} \mathrm{d}u_i \otimes \mathrm{d}u_i = g.$$

According to Theorem 3.1(ii), the standard metrics g of compact rank 1 symmetric spaces are critical metrics of the functionals λ_k, for any k such that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$. More generally, this is the case for all compact Riemannian homogeneous spaces with irreducible isotropy representation. Indeed, if $\{u_1, \ldots, u_d\}$ is an $L^2(g)$-orthonormal basis of $E_k(g)$, then the tensor $\sum_{i\le d} \mathrm{d}u_i \otimes \mathrm{d}u_i$ is invariant under the isometry group action which implies that it is proportional to g (Schur's Lemma).

In [14], we studied the notion of critical metrics of the trace of the heat kernel $Z_g(t) = \sum \mathrm{e}^{-\lambda_k(g)t}$, considered as a functional on the set of metrics of given volume. We obtain various characterizations of these critical metrics. An immediate consequence of Theorem 3.1 and [14, Theorem 2.2], is the following:

Corollary 3.1. *Let g be a Riemannian metric on M. If g is a critical metric of the trace of the heat kernel at any time $t > 0$, then g is a critical metric of the functional λ_k for all k such that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$.*

In particular, the flat metrics g_{sq} and g_{eq} on the 2-torus $\mathbb{T}^2$ associated with the square lattice $\mathbb{Z}^2$ and the equilateral lattice $\mathbb{Z}(1, 0) \oplus \mathbb{Z}(1/2, \sqrt{3}/2)$, respectively, are critical metrics of the functionals λ_k for all k such that $\lambda_k(g_{sq}) > \lambda_{k-1}(g_{sq})$ or $\lambda_k(g_{sq}) < \lambda_{k+1}(g_{sq})$ and such that $\lambda_k(g_{eq}) > \lambda_{k-1}(g_{eq})$ or $\lambda_k(g_{eq}) < \lambda_{k+1}(g_{eq})$ respectively (see [14]). Other examples of critical metrics can be obtained as Riemannian products of previous examples (see [14]).

As we noticed in the proof of Lemma 3.1, the condition $\sum_{i\le d} \mathrm{d}u_i \otimes \mathrm{d}u_i = g$, with $u_i \in E_k(g)$, implies that the map $u = (u_1, \ldots, u_d)$ is an isometric immersion of (M, g) into a $(d-1)$-dimensional sphere. In particular, the rank of u is at least n. Therefore we have the following:

Corollary 3.2. *If g is a critical metric of the functional λ_k, then*

$$\dim E_k(g) \ge \dim M + 1.$$

Moreover, the equality implies that (M, g) is isometric to a Euclidean sphere.

In the particular case where a metric g is a local maximizer of λ_k (that is $\lambda_k(g_t) \le \lambda_k(g)$ for any volume-preserving deformation $(g_t)_t$ of g), we have the additional necessary condition that $\lambda_k(g) = \lambda_{k+1}(g)$ (see Proposition 4.2 below). For a local minimizer, we have $\lambda_k(g) = \lambda_{k-1}(g)$. In particular, the functional λ_1 admits no local minimizing metric. This result was obtained by us in [13] using different arguments.

We end this section with the following result in the spirit of Berger's work [4]:

Corollary 3.3. *Let g be a Riemannian metric on M. Let $p \ge 1$ and $q \ge p$ be two natural integers such that*

$$\lambda_{p-1}(g) < \lambda_p(g) = \lambda_{p+1}(g) = \cdots = \lambda_q(g) < \lambda_{q+1}(g).$$

The metric g is critical for the functional $\sum_{i=p}^{q} \lambda_i$ if and only if there exists an $L^2(M, g)$-orthonormal basis $\{u_1, u_2, \ldots, u_m\}$ of $E_p(g)$ such that $\sum_{i=1}^{m} \mathrm{d}u_i \otimes \mathrm{d}u_i$ is proportional to g.

Proof. The multiplicity of $\lambda_p(g)$ is $m = q - p + 1$. Let $(g_t)_t$ be a volume-preserving analytic deformation of g and $h = \frac{\mathrm{d}}{\mathrm{d}t} g_t|_{t=0} \in S_0^2(M, g)$. Let $\Lambda_1(t), \ldots, \Lambda_m(t)$ and $v_1(t), \ldots, v_m(t)$ be the families of eigenvalues and orthonormal eigenfunctions of Δ_{g_t} depending analytically on t and such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_p(g)$, as in the proof of Theorem 2.1. For sufficiently small t, one has

$$\sum_{i=p}^{q} \lambda_i(g_t) = \sum_{i=1}^{m} \Lambda_i(t).$$

Hence, $\sum_{i=p}^{q} \lambda_i(g_t)$ is differentiable at $t = 0$ and one has (see the proof of Theorem 2.1 and Lemma 2.1) $\frac{\mathrm{d}}{\mathrm{d}t} \sum_{i=p}^{q} \lambda_i(g_t)|_{t=0} = \sum_{i=1}^{m} \Lambda_i'(0) = \sum_{i=1}^{m} Q_h(v_i)$, with $v_i := v_i(0)$. Therefore, g is critical for $\sum_{i=p}^{q} \lambda_i$ if and only if, for all $h \in S_0^2(M, g)$, $\sum_{i=1}^{m} Q_h(v_i) = 0$. As in the proof of Lemma 3.1, this last condition means that $\sum_{i\le m} \mathrm{d}v_i \otimes \mathrm{d}v_i$ is proportional to g. $\square$

4. Critical metrics of the eigenvalue functionals in a conformal class

Let M be a closed manifold of dimension $n \ge 2$. For any Riemannian metric g on M, we will denote by $C(g)$ the set of metrics which are conformal to g and have the same volume as g, i.e.

$$C(g) = \left\{\mathrm{e}^{\alpha} g;\ \alpha \in \mathcal{C}^{\infty}(M) \text{ and } \mathrm{vol}(\mathrm{e}^{\alpha} g) = \mathrm{vol}(g)\right\}.$$

Let k be a positive integer. The purpose of this section is to study critical metrics of the functional λ_k restricted to a conformal class $C(g)$.

Definition 4.1. A metric g is said to be critical for the functional λ_k restricted to $C(g)$ if, for any analytic deformation $\{g_t = e^{\alpha_t} g\} \subset C(g)$ with $g_0 = g$, we have

$$\frac{d}{dt}\lambda_k(g_t)\bigg|_{t=0^-} \times \frac{d}{dt}\lambda_k(g_t)\bigg|_{t=0^+} \leq 0.$$

In the sequel, we denote by $\mathcal{A}_0(M, g)$ the set of regular functions φ with zero mean on M, that is, $\int_M \varphi\, v_g = 0$. In the spirit of Propositions 3.1 and 3.2, we obtain in the conformal setting, the following:

Proposition 4.1. *Let g be a Riemannian metric on M.*

(i) *If g is a critical metric of the functional λ_k restricted to $C(g)$, then, for all $\varphi \in \mathcal{A}_0(M, g)$, the quadratic form*

$$q_\varphi(u) = \int_M \left(\lambda_k(g)u^2 - \frac{n-2}{n}|du|^2\right)\varphi\, v_g$$

is indefinite on $E_k(g)$.

(ii) *Assume that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$. The metric g is critical for the functional λ_k restricted to $C(g)$ if and only if, for all $\varphi \in \mathcal{A}_0(M, g)$, the quadratic form q_φ is indefinite on $E_k(g)$.*

Proof. (i) Let $\varphi \in \mathcal{A}_0(M, g)$. The conformal deformation of g given by

$$g_t := \left[\frac{\mathrm{vol}(g)}{\mathrm{vol}(e^{t\varphi} g)}\right]^{\frac{2}{n}} e^{t\varphi} g,$$

belongs to $C(g)$ and depends analytically on t with $\frac{d}{dt} g_t|_{t=0} = \varphi g$. Following the arguments of the proof of Proposition 3.1, we show that the criticality of g for λ_k restricted to $C(g)$ implies the indefiniteness of the quadratic form $Q_{\varphi g}$ on $E_k(g)$. Applying Lemma 2.1, we observe that $Q_{\varphi g} = -\frac{n}{2} q_\varphi$.

(ii) Let $g_t = e^{\alpha_t g} \in C(g)$ be an analytic deformation of g. Since $\mathrm{vol}(g_t)$ is constant with respect to t, the function $\varphi = \frac{d}{dt}\alpha_t|_{t=0}$ belongs $\mathcal{A}_0(M, g)$. Applying Theorem 2.1(iii) and (iv) and Lemma 2.1 with $h = \varphi g$, we get the result. □

Lemma 4.1. *Let g be a Riemannian metric on M. The two following conditions are equivalent:*

(i) *For all $\varphi \in \mathcal{A}_0(M, g)$, the quadratic form q_φ is indefinite on $E_k(g)$.*

(ii) *There exists a finite family $\{u_1, \ldots, u_d\} \subset E_k(g)$ of eigenfunctions associated with $\lambda_k(g)$ such that*

$$\sum_{i \leq d} u_i^2 = 1.$$

Proof. "(i) implies (ii)": We introduce the convex set

$$H = \left\{\sum_{i \in I}\left[\lambda_k(g)u_i^2 - \frac{n-2}{n}|du_i|^2\right];\, u_i \in E_k(g),\, I \subset \mathbb{N},\, I \text{ finite}\right\}.$$

Using the same arguments as in the proof of Lemma 3.1, we show that the constant function 1 belongs to H. Hence, there exist $u_1, \ldots, u_d \in E_k(g)$ such that

$$\sum_{i \leq d}\left(\lambda_k(g)u_i^2 - \frac{n-2}{n}|du_i|^2\right) = \frac{2}{n}\lambda_k(g).$$

For $n = 2$, we immediately get $\sum_{i \leq d} u_i^2 = 1$. For $n \geq 3$, we set $f = \sum_{i \leq d} u_i^2 - 1$ and get, after a straightforward calculation,

$$\frac{n-2}{4}\Delta_g f = -\lambda_k(g) f.$$

Thus, $f = 0$ and $\sum_{i \le d} u_i^2 = 1$.

"(ii) implies (i)": let $u_1, \ldots, u_d \in E_k(g)$ be such that $\sum_{i \le d} u_i^2 = 1$. One has

$$\sum_{i \le d} |du_i|^2 = -\frac{1}{2} \Delta_g \sum_{i \le d} u_i^2 + \lambda_k(g) \sum_{i \le d} u_i^2 = \lambda_k(g).$$

Therefore, for all $\varphi \in \mathcal{A}_0(M, g)$,

$$\sum_{i \le d} q_\varphi(u_i) = \frac{2}{n} \int_M \lambda_k(g) \varphi v_g = 0$$

which implies the indefiniteness of q_φ. □

Proposition 4.1 and Lemma 4.1 lead to the following:

Theorem 4.1. *Let g be a Riemannian metric on M.*

(i) *If g is a critical metric of the functional λ_k restricted to $C(g)$, then there exists a finite family $\{u_1, \ldots, u_d\} \subset E_k(g)$ of eigenfunctions associated with λ_k such that $\sum_{i \le d} u_i^2 = 1$.*

(ii) *Assume that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$. Then, g is critical for the functional λ_k restricted to $C(g)$ if and only if there exists a finite family $\{u_1, \ldots, u_d\} \subset E_k(g)$ of eigenfunctions associated with $\lambda_k(g)$ such that*

$$\sum_{i \le d} u_i^2 = 1.$$

The Riemannian metric g of any compact homogeneous Riemannian space (M, g) is a critical metric of the functional λ_k restricted to $C(g)$ for all k such that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$. Indeed, any $L^2(g)$-orthonormal basis $\{u_i\}_{i \le d}$ of $E_k(g)$ is such that $\sum_{i \le d} u_i^2$ is constant on M. In [14, Theorem 4.1], we proved that a metric g is critical for the trace of the heat kernel restricted to $C(g)$ if and only if its heat kernel K is constant on the diagonal of $M \times M$. This last condition implies that, for all $k \in \mathbb{N}^*$, any $L^2(g)$-orthonormal basis $\{u_i\}_{i \le d}$ of $E_k(g)$ is such that $\sum_{i \le d} u_i^2$ is constant on M. Hence, we have the following:

Corollary 4.1. *Let g be a Riemannian metric on M and let K be the heat kernel of (M, g). Assume that, for all $t > 0$, the function $x \in M \longmapsto K(t, x, x)$ is constant; then the metric g is critical for the functional λ_k restricted to $C(g)$ for all k such that $\lambda_k(g) > \lambda_{k-1}(g)$ or $\lambda_k(g) < \lambda_{k+1}(g)$.*

An immediate consequence of Theorem 4.1 is the following:

Corollary 4.2. *If g is a critical metric of the functional λ_k restricted to $C(g)$, then $\lambda_k(g)$ is a degenerate eigenvalue, that is*

$$\dim E_k(g) \ge 2.$$

This last condition means that at least one of the following holds: $\lambda_k(g) = \lambda_{k-1}(g)$ or $\lambda_k(g) = \lambda_{k+1}(g)$. In the case when g is a local maximizer or a local minimizer, we have the following more precise result:

Proposition 4.2. (i) *If g is a local maximizer of the functional λ_k restricted to $C(g)$, then $\lambda_k(g) = \lambda_{k+1}(g)$.*
(ii) *If g is a local minimizer of the functional λ_k restricted to $C(g)$, then $\lambda_k(g) = \lambda_{k-1}(g)$.*

Proof. Assume that g is a local maximizer and that $\lambda_k(g) < \lambda_{k+1}(g)$. Let $\varphi \in \mathcal{A}_0(M, g)$ and let $g_t = e^{\alpha_t} g \in C(g)$ be a volume-preserving analytic deformation of g such that $\frac{d}{dt} g_t|_{t=0} = \varphi g$. Denote by $\Lambda_1(t), \ldots, \Lambda_m(t)$, with $m = \dim E_k(g)$, the associated family of eigenvalues of Δ_{g_t}, depending analytically on t and such that $\Lambda_1(0) = \cdots = \Lambda_m(0) = \lambda_k(g)$ (see the proof of Theorem 2.1). For continuity reasons, we have, for sufficiently small t and all $i \le m$,

$$\Lambda_i(t) < \lambda_{k+1}(g_t).$$

Hence, for all $i \le m$ and all t sufficiently small,

$$\Lambda_i(t) \le \lambda_k(t) \le \lambda_k(g) = \Lambda_i(0).$$

Consequently, $\Lambda_i'(0) = 0$ for all $i \le m$. Since $\Lambda_1'(0), \dots, \Lambda_m'(0)$ are eigenvalues of the operator $\Pi_k \circ \Delta'$ (by Theorem 2.1), this operator is identically zero on $E_k(g)$. Applying Lemma 2.1, we deduce that, for all $\varphi \in \mathcal{A}_0(M, g)$, $Q_{\varphi g} \equiv 0$ on $E_k(g)$. Thus, for all $u \in E_k(g)$, there exists a constant c such that

$$|\mathrm{d}u|^2 + \frac{n}{4}\Delta_g u^2 = c.$$

Integrating, we get $c = \frac{\lambda_k(g)}{\mathrm{vol}(g)} \int_M u^2 v_g$. Since $\Delta_g u^2 = 2(\lambda_k u^2 - |\mathrm{d}u|^2)$, we obtain

$$\frac{n}{2}u^2 - \frac{n-2}{2\lambda_k(g)}|\mathrm{d}u|^2 = \frac{1}{\mathrm{vol}(g)} \int_M u^2 v_g.$$

Let $x_0 \in M$ be a point where u^2 achieves its maximum. At x_0, we have $\mathrm{d}u(x_0) = 0$ and, then,

$$\frac{n}{2}\max u^2 = \frac{n}{2}u^2(x_0) = \frac{1}{\mathrm{vol}(g)} \int_M u^2 v_g$$

which leads to a contradiction (since u is not constant and $\frac{n}{2} \ge 1$).

A similar proof works for (ii). □

In [6], Colbois and the first author proved that

$$\sup_{g' \in C(g)} \lambda_{k+1}(g')^{\frac{n}{2}} - \sup_{g' \in C(g)} \lambda_k(g')^{\frac{n}{2}} \ge n^{\frac{n}{2}}\omega_n, \tag{6}$$

where ω_n is the volume of the unit Euclidean sphere of dimension n.

An immediate consequence of this result and Proposition 4.2 is the following:

Corollary 4.3. *Let g be a Riemannian metric on M. Assume that g maximizes the functional λ_k restricted to $C(g)$, that is*

$$\lambda_k(g) = \sup_{g' \in C(g)} \lambda_k(g').$$

Then g maximizes neither λ_{k+1} nor λ_{k-1} (for $k \ge 2$) on $C(g)$.

More precisely, if g maximizes λ_k on $C(g)$, then, using Proposition 4.2 and (6),

$$\lambda_{k+1}(g)^{\frac{n}{2}} \le \sup_{g' \in C(g)} \lambda_{k+1}(g')^{\frac{n}{2}} - n^{\frac{n}{2}}\omega_n.$$

Finally we have the following conformal version of Corollary 3.3:

Corollary 4.4. *Let g be a Riemannian metric on M. Let $p \ge 1$ and $q \ge p$ be two natural integers such that*

$$\lambda_{p-1}(g) < \lambda_p(g) = \lambda_{p+1}(g) = \cdots = \lambda_q(g) < \lambda_{q+1}(g).$$

The metric g is critical for the functional $\sum_{i=p}^{q} \lambda_i$ restricted to $C(g)$ if and only if there exists an $L^2(M, g)$-orthonormal basis $\{u_1, u_2, \dots, u_m\}$ of $E_p(g)$ such that $\sum_{i=1}^{m} u_i^2$ is constant on M.

5. Critical metrics of the eigenvalue ratios functionals

Let M be a closed manifold of dimension $n \ge 2$ and let k be a positive integer. This section deals with the functional $g \longmapsto \frac{\lambda_{k+1}(g)}{\lambda_k(g)}$. This functional is invariant under scaling, so it is not necessary to fix the volume of metrics under consideration. If $(g_t)_t$ is an analytic deformation of a metric g, then $t \longmapsto \frac{\lambda_{k+1}(g_t)}{\lambda_k(g_t)}$ admits left and right derivatives at $t = 0$ (Theorem 2.1).

Definition 5.1. (i) A metric g is said to be critical for the ratio $\frac{\lambda_{k+1}}{\lambda_k}$ if, for any analytic deformation (g_t) of g, the left and right derivatives of $\frac{\lambda_{k+1}(g_t)}{\lambda_k(g_t)}$ at $t = 0$ have opposite signs.

(ii) The metric g is said to be critical for the ratio functional $\frac{\lambda_{k+1}}{\lambda_k}$ restricted to the conformal class $C(g)$ if the condition above holds for any conformal analytic deformation $g_t = e^{\alpha_t} g$ of g.

Let g be a Riemannian metric on M. For any covariant 2-tensor $h \in S^2(M)$, we introduce the following operator:

$$\tilde{P}_{k,h} : E_k(g) \otimes E_{k+1}(g) \longrightarrow E_k(g) \otimes E_{k+1}(g)$$

defined by

$$\tilde{P}_{k,h} = \lambda_{k+1}(g) P_{h,k} \otimes Id_{E_{k+1}(g)} - \lambda_k(g) Id_{E_k(g)} \otimes P_{k+1,h},$$

where $P_{k,h}$ is defined in Lemma 2.1. The quadratic form naturally associated with $\tilde{P}_{k,h}$ is denoted by $\tilde{Q}_{k,h}$ and is given, for all $u \in E_k(g)$ and $v \in E_{k+1}(g)$, by

$$\tilde{Q}_{k,h}(u \otimes v) = \lambda_{k+1}(g) \|v\|^2_{L^2(g)} Q_h(u) - \lambda_k(g) \|u\|^2_{L^2(g)} Q_h(v),$$

where $Q_h(u) = -\int_M (\mathrm{d}u \otimes \mathrm{d}u + \frac{1}{4}\Delta_g u^2 g, h) v_g$.

Of course, if $\lambda_{k+1}(g) = \lambda_k(g)$, then g is a global minimizer of the ratio $\frac{\lambda_{k+1}}{\lambda_k}$. Notice that, thanks to Colin de Verdière's result [7], for all $k \geq 1$, any closed manifold M carries a metric g such that $\lambda_{k+1}(g) = \lambda_k(g)$. A general characterization of critical metrics of $\frac{\lambda_{k+1}}{\lambda_k}$ is given in what follows.

Proposition 5.1. *A Riemannian metric g on M is critical for the functional $\frac{\lambda_{k+1}}{\lambda_k}$ if and only if, for all $h \in S^2(M)$, the quadratic form $\tilde{Q}_{k,h}$ is indefinite on $E_k(g) \otimes E_{k+1}(g)$.*

Proof. The case where $\lambda_{k+1}(g) = \lambda_k(g)$ is obvious ($\tilde{Q}_{k,h}(u \otimes u) = 0$). Assume that $\lambda_{k+1}(g) > \lambda_k(g)$ and let $(g_t)_t$ be an analytic deformation of g. From Theorem 2.1, $\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)|_{t=0^-}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)|_{t=0^+}$ are the least and the greatest eigenvalues of $P_{k,h}$ on $E_k(g)$ respectively.

Similarly, $\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)|_{t=0^-}$ and $\frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)|_{t=0^+}$ are the greatest and the least eigenvalues of P_{k+1} on $E_k(g)$. Therefore,

$$\lambda_k(g)^2 \frac{\mathrm{d}}{\mathrm{d}t} \frac{\lambda_{k+1}(g_t)}{\lambda_k(g_t)}\bigg|_{t=0^-} = \left[\lambda_k(g) \frac{\mathrm{d}}{\mathrm{d}t}\lambda_{k+1}(g_t)\bigg|_{t=0^-} - \lambda_{k+1}(g) \frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\bigg|_{t=0^-}\right]$$

is the greatest eigenvalue of $\tilde{P}_{k,h}$ on $E_k(g) \otimes E_{k+1}(g)$, and

$$\lambda_k(g)^2 \frac{\mathrm{d}}{\mathrm{d}t} \frac{\lambda_{k+1}(g_t)}{\lambda_k(g_t)}\bigg|_{t=0^+} = \left[\lambda_k(g) \frac{\mathrm{d}}{\mathrm{d}t}\lambda_{k+1}(g_t)\bigg|_{t=0^+} - \lambda_{k+1}(g) \frac{\mathrm{d}}{\mathrm{d}t}\lambda_k(g_t)\bigg|_{t=0^+}\right]$$

is the least eigenvalue of $\tilde{P}_{k,h}$ on $E_k(g) \otimes E_{k+1}(g)$. Hence, the criticality of g for λ_{k+1}/λ_k is equivalent to the fact that $\tilde{P}_{k,h}$ admits eigenvalues of both signs, which is equivalent to the indefiniteness of $\tilde{Q}_{k,h}$. □

Lemma 5.1. *Let g be a Riemannian metric on M. The two following conditions are equivalent:*

(i) *For all $h \in S^2(M)$, the quadratic form $\tilde{Q}_{k,h}$ is indefinite on $E_k(g) \otimes E_{k+1}(g)$.*
(ii) *There exist two finite families $\{u_1, \ldots, u_p\} \subset E_k(g)$ and $\{v_1, \ldots, v_q\} \subset E_{k+1}(g)$ of eigenfunctions associated with $\lambda_k(g)$ and $\lambda_{k+1}(g)$ respectively, such that*

$$\sum_{i \leq p} \left(\mathrm{d}u_i \otimes \mathrm{d}u_i + \frac{1}{4}\Delta_g u_i^2 g\right) = \sum_{j \leq q} \left(\mathrm{d}v_j \otimes \mathrm{d}v_j + \frac{1}{4}\Delta_g v_j^2 g\right).$$

Proof. "(i) implies (ii)": Let us introduce the following two convex cones:

$$K_1 = \left\{\sum_{i \in I} \left(\mathrm{d}u_i \otimes \mathrm{d}u_i + \frac{1}{4}\Delta_g u_i^2 g\right); u_i \in E_k(g), I \subset \mathbb{N}, I \text{ finite}\right\} \subset S^2(M)$$

and

$$K_2=\left\{\sum_{j\in J}\left(\mathrm{d}v_j\otimes\mathrm{d}v_j+\frac{1}{4}\Delta_g v_j^2 g\right);\, v_j\in E_{k+1}(g),\, J\subset\mathbb{N},\, J\text{ finite}\right\}\subset S^2(M).$$

It suffices to prove that K_1 and K_2 have a non-trivial intersection. Indeed, otherwise, applying classical separation theorems, we show the existence of a 2-tensor $h\in S^2(M)$ such that, for all $T_1\in K_1$, $T_1\neq 0$,

$$\int_M (T_1,h)v_g>0$$

and, for all $T_2\in K_2$,

$$\int_M (T_2,h)v_g\leq 0.$$

Therefore, for all $u\in E_k(g)$ and all $v\in E_{k+1}(g)$, with $u\neq 0$ and $v\neq 0$, one has $Q_h(u)<0$, $Q_h(v)\geq 0$ and

$$\begin{aligned}\tilde{Q}_{k,h}(u\otimes v)&=\lambda_{k+1}(g)\,\|v\|^2_{L^2(g)}\,Q_h(u)-\lambda_k(g)\,\|u\|^2_{L^2(g)}\,Q_h(v)\\&\leq\lambda_{k+1}(g)\,\|v\|^2_{L^2(g)}\,Q_h(u)<0,\end{aligned}$$

which implies that $\tilde{Q}_{k,h}$ is negative definite on $E_k(g)\otimes E_{k+1}(g)$.

"(ii) implies (i)": Let $\{u_i\}_{i\leq p}$ and $\{v_j\}_{j\leq q}$ be as in (ii). From the identity in (ii), we get, after taking the trace and integrating,

$$\sum_{i\leq p}\int_M|\mathrm{d}u_i|^2v_g=\sum_{j\leq q}\int_M|\mathrm{d}v_j|^2v_g,$$

which gives

$$\lambda_k(g)\sum_{i\leq p}\|u_i\|^2_{L^2(g)}=\lambda_{k+1}(g)\sum_{j\leq q}\left\|v_j\right\|^2_{L^2(g)}.$$

Now,

$$\sum_{i,j}\tilde{Q}_{k,h}(u_i\otimes v_j)=\sum_{i,j}\lambda_{k+1}(g)\left\|v_j\right\|^2_{L^2(g)}Q_h(u_i)-\lambda_k(g)\,\|u_i\|^2_{L^2(g)}\,Q_h(v_j).$$

Assumption (ii) implies that $\sum_{i\leq p}Q_h(u_i)=\sum_{j\leq q}Q_h(v_j)$. Therefore,

$$\sum_{i,j}\tilde{Q}_{k,h}(u_i\otimes v_j)=\left(\sum_{j\leq q}\lambda_{k+1}(g)\left\|v_j\right\|^2_{L^2(g)}-\sum_{i\leq p}\lambda_k(g)\,\|u_i\|^2_{L^2(g)}\right)\sum_{i\leq p}Q_h(u_i)=0.$$

Hence, $\tilde{Q}_{k,h}$ is indefinite on $E_k(g)\otimes E_{k+1}(g)$. □

Consequently, we have proved the following:

Theorem 5.1. *A metric g on M is critical for the functional* $\frac{\lambda_{k+1}}{\lambda_k}$ *if and only if there exist two families* $\{u_1,\ldots,u_p\}\subset E_k(g)$ *and* $\{v_1,\ldots,v_q\}\subset E_{k+1}(g)$ *of eigenfunctions associated with* $\lambda_k(g)$ *and* $\lambda_{k+1}(g)$*, respectively, such that*

$$\sum_{i\leq p}\mathrm{d}u_i\otimes\mathrm{d}u_i-\sum_{j\leq q}\mathrm{d}v_j\otimes\mathrm{d}v_j=\alpha g \tag{7}$$

for some $\alpha\in\mathcal{C}^\infty(M)$*, and*

$$\lambda_k(g)\sum_{i\leq p}u_i^2-\lambda_{k+1}(g)\sum_{j\leq q}v_j^2=\frac{n-2}{n}\left(\sum_{i\leq p}|\mathrm{d}u_i|^2-\sum_{j\leq q}|\mathrm{d}v_j|^2\right). \tag{8}$$

Indeed, a straightforward calculation shows that the two Eqs. (7) and (8) are equivalent to the condition (ii) of Lemma 5.1.

Corollary 5.1. *If g is a critical metric of the functional $\frac{\lambda_{k+1}}{\lambda_k}$, with $\lambda_{k+1}(g) \neq \lambda_k(g)$, then*

$$\min\{\dim E_k(g), \dim E_{k+1}(g)\} \geq 2.$$

Proof. Let $\{u_i\}_{i\leq p} \subset E_k(g)$ and $\{v_j\}_{j\leq q} \subset E_{k+1}(g)$ be two families of eigenfunctions satisfying (7) and (8) above. Taking the trace in (7) and using (8) we get

$$\alpha = \frac{1}{n}\left(\sum_{i\leq p} |\mathrm{d}u_i|^2 - \sum_{j\leq q} |\mathrm{d}v_j|^2\right) = \frac{1}{4}\Delta_g\left(\sum_{j\leq q} v_j^2 - \sum_{i\leq p} u_i^2\right). \tag{9}$$

Assume that $\alpha = 0$. Using (8) and (9), we deduce that both $\sum_{i\leq p} u_i^2$ and $\sum_{j\leq q} v_j^2$ are constant on M. Since the u_i's and the v_j's are not constant, we get the result. Assume now $\alpha \neq 0$. Since $\int_M \alpha\, v_g = 0$ (see (9)), the function α takes both positive and negative values. Let $x \in M$ be such that $\alpha(x) > 0$. From (7), the quadratic form $\sum_{i\leq p} \mathrm{d}u_i \otimes \mathrm{d}u_i$ is clearly positive definite on T_xM. Hence, the family $\{\mathrm{d}u_i\}$ has maximal rank at x. This shows that $\dim E_k(g) \geq n$. At a point $x \in M$ where $\alpha(x) < 0$, the quadratic form $\sum_{j\leq q} \mathrm{d}v_j \otimes \mathrm{d}v_j$ is positive definite on T_xM and, then, $\dim E_{k+1}(g) \geq n$. □

When we deal with critical metrics of the functional $\frac{\lambda_{k+1}}{\lambda_k}$ restricted to $C(g)$, only tensors of the form $h = \varphi\, g$, with $\varphi \in \mathcal{C}^\infty(M)$, are involved. The corresponding quadratic forms on $E_k(g) \otimes E_{k+1}(g)$ are given by

$$\tilde{q}_{k,\varphi}(u \otimes v) = \lambda_{k+1}(g)\, \|v\|^2_{L^2(g)}\, q_\varphi(u) - \lambda_k(g)\, \|u\|^2_{L^2(g)}\, q_\varphi(v).$$

Following the steps of the proof of Proposition 5.1, we can show that:

Proposition 5.2. *A Riemannian metric g on M is critical for the functional $\frac{\lambda_{k+1}}{\lambda_k}$ restricted to $C(g)$ if and only if, for all $\varphi \in \mathcal{C}^\infty(M)$, the quadratic form $\tilde{q}_{k,\varphi}$ is indefinite on $E_k(g) \otimes E_{k+1}(g)$.*

Replacing the convex cones K_1 and K_2 in the proof of Lemma 5.1 by

$$H_1 = \left\{\sum_{i\in I}\left(\lambda_k(g)u_i^2 - \frac{n-2}{n}|\mathrm{d}u_i|^2\right); u_i \in E_k(g), I \subset \mathbb{N}, I \text{ finite}\right\} \subset L^2(M, g)$$

and

$$H_2 = \left\{\sum_{j\in J}\left(\lambda_{k+1}(g)v_j^2 - \frac{n-2}{n}|\mathrm{d}v_j|^2\right); v_j \in E_{k+1}(g), J \subset \mathbb{N}, J \text{ finite}\right\} \subset L^2(M, g),$$

we can show, by the same arguments, that the indefiniteness of $\tilde{q}_{k,\varphi}$ for all $\varphi \in \mathcal{C}^\infty(M)$ is equivalent to the fact that H_1 and H_2 have a non-trivial intersection. Therefore, one has:

Theorem 5.2. *A Riemannian metric g on M is critical for the functional $\frac{\lambda_{k+1}}{\lambda_k}$ restricted to $C(g)$ if and only if there exist two families $\{u_1, \ldots, u_p\} \subset E_k(g)$ and $\{v_1, \ldots, v_q\} \subset E_{k+1}(g)$ of eigenfunctions associated with $\lambda_k(g)$ and $\lambda_{k+1}(g)$, respectively, such that*

$$\lambda_k(g)\sum_{i\leq p} u_i^2 - \lambda_{k+1}(g)\sum_{j\leq q} v_j^2 = \frac{n-2}{n}\left(\sum_{i\leq p}|\mathrm{d}u_i|^2 - \sum_{j\leq q}|\mathrm{d}v_j|^2\right).$$

Remark 5.1. In dimension 2, the condition of Theorem 5.2 amounts to

$$\sum_{j\leq q} v_j^2 = \frac{\lambda_k(g)}{\lambda_{k+1}(g)}\sum_{i\leq p} u_i^2.$$

It is clear that in this case, if $\lambda_{k+1}(g) \neq \lambda_k(g)$, then at least one of the eigenvalues $\lambda_k(g)$ and $\lambda_{k+1}(g)$ is degenerate.

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